# 13.1-13.4 Curves in 3D

*Example:*

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$$

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ tangent vector

$$s(t) = \int_0^t |\vec{r}'(u)| du = \text{distance}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \text{unit tangent}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \text{principal unit normal}$$

$$K = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \text{curvature}$$

Find $K$.

$$\vec{r}' = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2\sin(t), & 2\cos(t), & 0 \end{vmatrix}$$

$$\vec{r}'' = \begin{vmatrix} -2\cos(t), & -2\sin(t), & 0 \end{vmatrix}$$

$$\vec{r}' \times \vec{r}'' = (0 - 0)\vec{\imath} - (0 - 0)\vec{\jmath} + (\overbrace{4\sin^2(t) - -4\cos^2(t)})\vec{k}$$

$$= \langle 0, 0, 4 \rangle$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{0^2 + 0^2 + 4^2} = 4$$

$$|\vec{r}'| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4} = 2$$

$$k = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{4}{2^3} = \frac{4}{8} = \frac{1}{2}$$

CONSTANT CURVATURE OF $\frac{1}{2}$

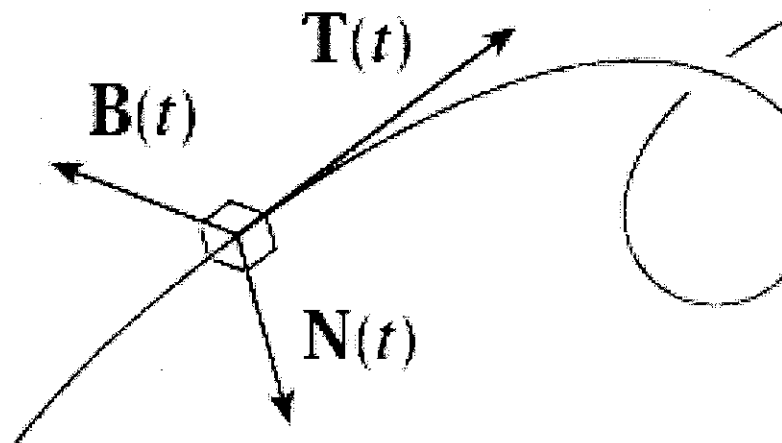$$\frac{1}{k} = \text{RADIUS OF CURVATURE} = 2$$

*Note*:

$T$ and $T'$ are always orthogonal.

*Proof*:

Since $T \cdot T = |T|^2 = 1$, we can differentiate both sides to get
$$T' \cdot T + T \cdot T' = 0.$$
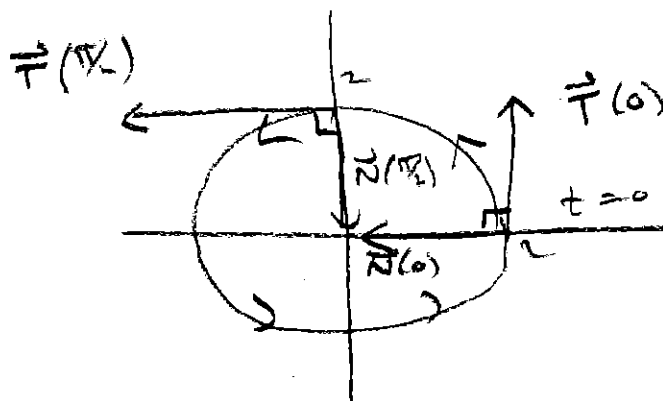
So $2T \cdot T' = 0$.

Thus, $T \cdot T' = 0$. (QED)

$\vec{T}(0) = \langle 0, 1, 0 \rangle$

$\vec{N}(0) = \langle -1, 0, 0 \rangle$

$\vec{T}(\pi/2) = \langle -1, 0, 0 \rangle$

$\vec{N}(\pi/2) = \langle 0, -1, 0 \rangle$

$T(t)$

$B(t)$

$N(t)$

LAST EXAMPLE

$\vec{T}(\pi/2)$

$\vec{N}(\pi/2)$

$\vec{T}(0)$

$\vec{N}(0)$

$t = 0$

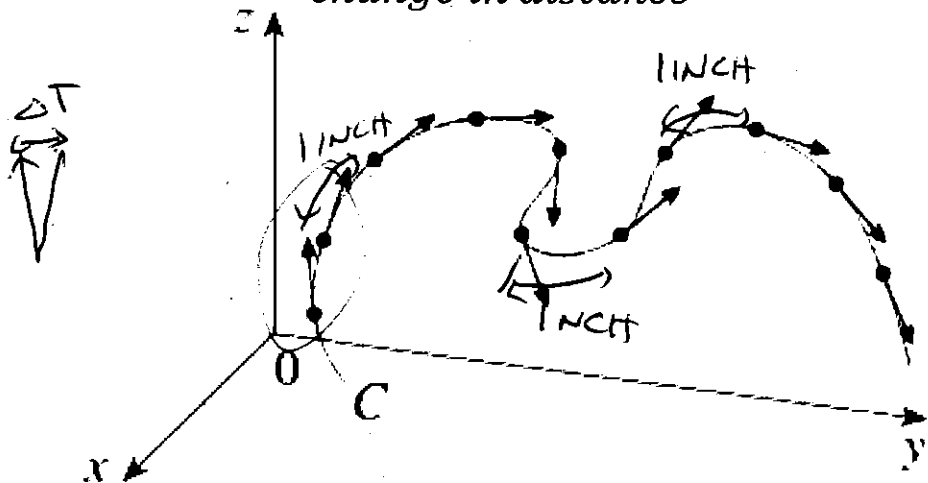$k = \frac{1}{2}$

"RADIUS OF CURVATURE"

$= \dfrac{1}{k} = 2$

IT IS!

"CURVING LIKE A CIRCLE OF RADIUS 2"

## Curvature

The **curvature** at a point, $K$, is a measure of how quickly a curve is changing direction at that point.

$$K = \frac{change\ in\ direction}{change\ in\ distance}$$



Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left| \frac{\overrightarrow{T_2} - \overrightarrow{T_1}}{"one\ inch"} \right| = \left| \frac{\Delta \overrightarrow{T}}{\Delta s} \right|$$

So we define:

$$K = \left| \frac{d\overrightarrow{T}}{ds} \right|.$$

## Computation Notes

*(see my 13.3 Notes/book for the proof)*

1st shortcut:

$$K(t) = \left| \frac{d\overrightarrow{T}}{ds} \right| = \left| \frac{d\overrightarrow{T}/dt}{ds/dt} \right| = \frac{|\overrightarrow{T}'(t)|}{|\overrightarrow{r}'(t)|}$$

2nd shortcut

$$K(t) = \left| \frac{d\overrightarrow{T}}{ds} \right| = \frac{|\overrightarrow{T}'(t)|}{|\overrightarrow{r}'(t)|} = \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3}$$

*Aside:*

The *radius of curvature* is the radius of the circle that would best fit the curve at the given point.

$$radius\ of\ curvature = \frac{1}{K}$$

## 2D Curvature

To find curvature for, $y = f(x)$,
We form the 3D vector function

$$r(x) = \langle x, f(x), 0 \rangle.$$

so $r'(x) = \langle 1, f'(x), 0 \rangle$ and

$r''(x) = \langle 0, f''(x), 0 \rangle$

$|r'(x)| = \sqrt{1 + (f'(x))^2}$

$r' \times r'' = \langle 0, 0, f''(x) \rangle$

Thus,

$$K(x) = \frac{|r' \times r''|}{|r'|^3} = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}$$

*Example:* $f(x) = x^2$
At what point $(x, y, z)$ is the
curvature maximum?

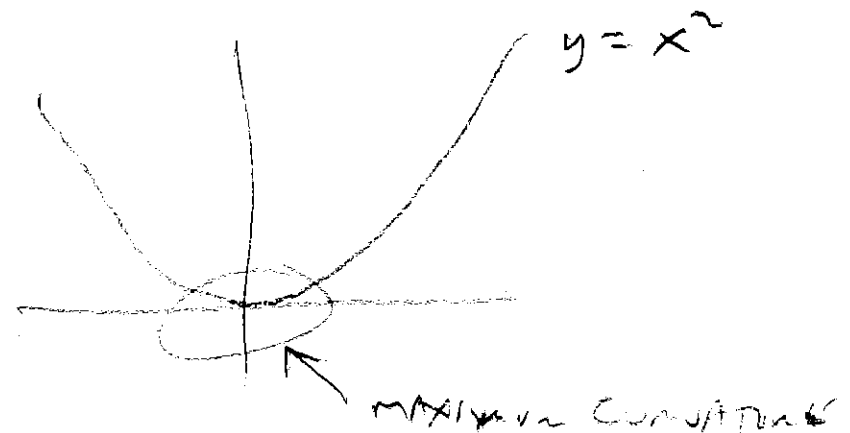$f'(x) = 2x$

$f''(x) = 2$

$$k(x) = \frac{2}{\left(1 + (2x)^2\right)^{3/2}} \qquad \leftarrow \text{MAXIMIZE!!!}$$

$k'(x) = 2\left(1 + 4x^2\right)^{-3/2}$

$k'(x) = -3\left(1 + 4x^2\right)^{-5/2} \cdot 8x \overset{?}{=} 0$

$x = 0 \quad \leftarrow \text{critical \#}$

$k(0) = 2 = \text{MAXIMUM}$



$y = x^2$

MAXIMUM CURVATURE

## 13.4 Position, Velocity, Acceleration

If $t$ = *time* and position is given by
$$r(t) = \langle x(t), y(t), z(t) \rangle$$
then

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$
$$= \frac{\text{change in position}}{\text{change in time}}$$
$$= \text{velocity} = v(t)$$

$$|r'(t)| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed}$$

$$r''(t) = \lim_{h \to 0} \frac{r'(t+h) - r'(t)}{h}$$
$$= \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \text{acceleration} = a(t)$$

Let $t$ be **time in seconds** and assume the position of an object (in **feet**) is given by
$$r(t) = \langle t, 2e^{-t}, 0 \rangle$$
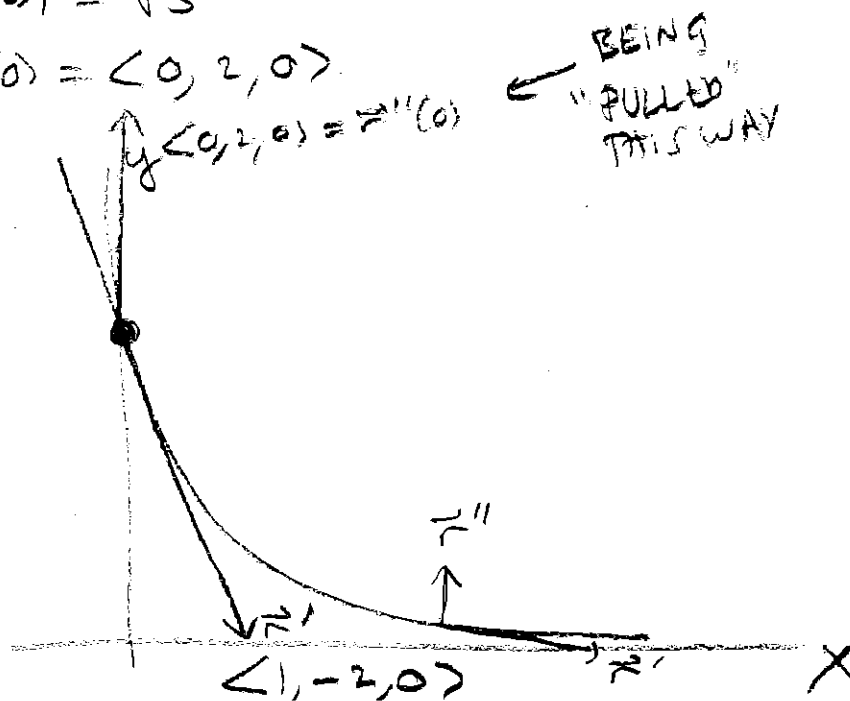Compute

1. $r'(t)$, $|r'(t)|$, and $r''(t)$.
2. $r'(0)$, $|r'(0)|$, and $r''(0)$.

$$\vec{r}'(t) = \langle 1, -2e^{-t}, 0 \rangle$$
$$|\vec{r}'(t)| = \sqrt{1 + 4e^{-2t}}$$
$$\vec{r}''(t) = \langle 0, 2e^{-t}, 0 \rangle$$

$$\vec{r}'(0) = \langle 1, -2, 0 \rangle$$
$$|\vec{r}'(0)| = \sqrt{5}$$
$$\vec{r}''(0) = \langle 0, 2, 0 \rangle$$

$\langle 0, 2, 0 \rangle = \vec{r}''(0)$

BEING "PULLED" THIS WAY

$\langle 0, 2, 0 \rangle$

$\vec{r}''$

$\vec{r}'$

$\langle 1, -2, 0 \rangle$

$\vec{r}$

X

## HUGE application:
## Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$F = m \cdot a \text{ , so}$$

$$a = \frac{1}{m} \cdot F$$

If $F = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant)

If $F \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve a differential equation) to find velocity and position.

That is how we can model ALL motion problems!

*HW Example:*

An object of mass 10 kg is being acted on by the force $F = \langle 130t, 10e^t, 10e^{-t} \rangle$. You are given

$$v(0) = \langle 0,0,1 \rangle \text{ and } r(0) = \langle 0,1,1 \rangle.$$

Find the position function.

$$\vec{a}(t) = \frac{1}{m}\vec{F} = \langle 13t, e^t, e^{-t} \rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$= \langle \tfrac{13}{2}t^2 + c_1, \; e^t + c_2, \; -e^{-t} + c_3 \rangle$$

$$\vec{v}(0) = \langle 0,0,1 \rangle \Rightarrow \langle c_1, 1+c_2, -1+c_3 \rangle = \langle 0,0,1 \rangle$$

$$c_1 = 0, \quad c_2 = -1, \quad c_3 = 2$$

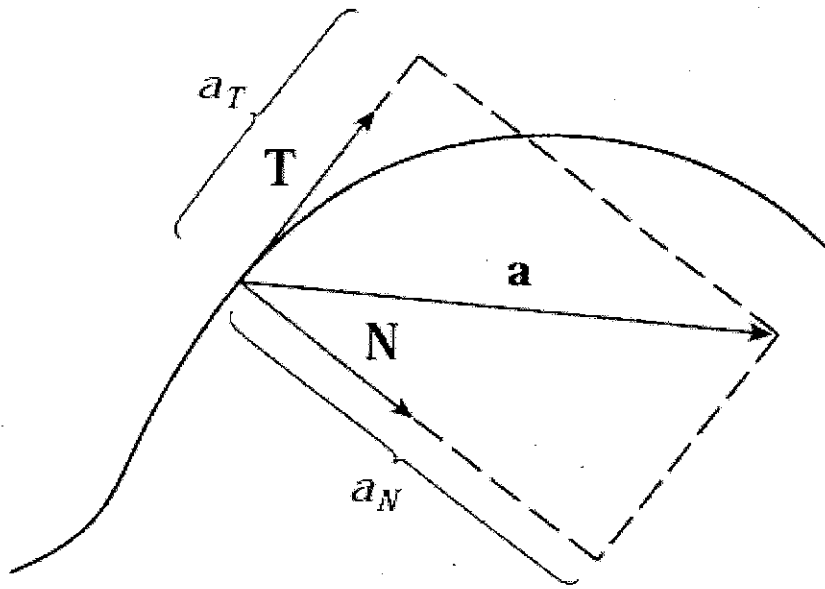$$\vec{v}(t) = \langle \tfrac{13}{2}t^2, \; e^t - 1, \; -e^{-t} + 2 \rangle$$

$$\vec{r}(t) = \langle \tfrac{13}{6}t^3 + d_1, \; e^t - t + d_2, \; e^{-t} + 2t + d_3 \rangle$$

$$\vec{r}(0) = \langle 0,1,1 \rangle \Rightarrow \langle d_1, 1+d_2, 1+d_3 \rangle = \langle 0,1,1 \rangle$$

$$d_1 = 0, \quad d_2 = 0, \quad d_3 = 0$$

$$\boxed{\vec{r}(t) = \langle \tfrac{13}{6}t^3, \; e^t - t, \; e^{-t} + 2t \rangle}$$

**Measuring and describing acceleration**



Recall: $\text{comp}_b(a) = \dfrac{a \cdot b}{b} = \text{length}$.

We define the tangential and normal components of acceleration by:

$a_T = \text{comp}_T(a) = a \cdot T$ = tangential

$a_N = \text{comp}_N(a) = a \cdot N$ = normal

For computing use,

$$a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|} \quad \text{and} \quad a_T = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|}$$

For interpreting use,

$a_T = v' = \dfrac{d}{dt}|r'(t)| =$ "deriv. of speed"

$a_N = kv^2 =$ curvature $\cdot$ (speed)$^2$

*Example:*

$$\vec{r}(t) = < \cos(t), \sin(t), t >$$

Find the tangential and normal components of acceleration.

$$\vec{r}'(t) = < -\sin(t), \cos(t), 1 >$$

$$|\vec{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$\vec{r}''(t) = < -\cos(t), -\sin(t), 0 >$$

$$\vec{r}' \cdot \vec{r}'' = \cos(t)\sin(t) - \cos(t)\sin(t) + 0 = 0$$

$$\Rightarrow \quad a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|} = 0 \iff \text{CONSTANT SPEED}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = (0 - -\sin(t))\vec{\imath} - (0 - -\cos(t))\vec{\jmath} + (\sin^2(t) - -\cos^2(t))\vec{k}$$

$$= < \sin(t), -\cos(t), 1 > \qquad \text{chain}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$a_N = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \qquad = K(\text{speed})^2 \Rightarrow K = \frac{1}{2}$$

RADIUS OF
CURVATURE = 2

*Deriving interpretations:*

Note that: $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$

Let $v(t) = |\vec{v}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \dfrac{\vec{v}(t)}{v(t)}$ implies $\vec{v} = v\vec{T}$.

2. $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{T}'|}{v(t)}$ implies $|\vec{T}'| = \kappa v$.

3. $\vec{\boldsymbol{N}}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \dfrac{\vec{T}'}{\kappa v}$ implies $\vec{T}' = \kappa v \vec{N}$.

Differentiating the first fact above gives

$$\vec{a} = \vec{v}' = v'\vec{T} + v\vec{T}', \text{ so}$$
$$\vec{a} = \vec{v}' = v'\vec{T} + kv^2\vec{N}.$$

*Conclusion:*

$$a_T = v' = \frac{d}{dt}|r'(t)| = \text{"deriv. of speed"}$$
$$a_N = kv^2 = \text{curvature} \cdot (\text{speed})^2$$